BCS-BEC model of high-T_c superconductivity in layered cuprates with unconventional pairing

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High-T_c superconductivity in layered cuprates is described in a BCS-BEC formalism with linearly-dispersive s- and d-wave Cooper pairs moving in quasi-2D finite-width layers about the CuO_2 planes. This yields a closed formula for T_c determined by the layer width, the Debye frequency, the pairing energy, and the in-plane penetration depth. The new formula reasonably reproduces empirical values of superconducting T_cs for seven different compounds among the $LSCO$, $YBCO$, $BSCCO$ and $TBCCO$ layered cuprates.

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I. INTRODUCTION

It seems to be well-established that central to high-T_c superconductivity (HTSC) is the layered two-dimensional (2D) structure of copper oxides and that superconducting pairing occurs mainly on the CuO_2 planes. However, the precise nature of the pairing is still the subject of intense research. Recent experiments based on angle-resolved photon emission spectroscopy (ARPES) of underdoped cuprates suggest that bound fermion Cooper pairs (CPs) form already at and below temperatures higher than the critical transition temperature T_c [1–4]. Furthermore, ARPES studies of the electron spectral function of optimally doped Bi2212 samples show that the magnitude of the isotope effect correlates with the superconducting gap [5], thus suggesting a role of lattice phonons in the superconducting pairing. ARPES data also suggest that the energy gap Δ (a measure of the energy needed to break a CP) displays an unconventional $d_{x^2-y^2}$ orbital pairing symmetry, with a functional dependence $\Delta = \Delta_0 \cos 2\theta$ where $\theta = \tan^{-1} K_y/K_x$ is the angle between the total or center-of-mass momentum (CMM) $\hbar K = (\hbar K_x, \hbar K_y)$ of paired electrons in the CuO_2 plane and the a- (or x-) axis while Δ_0 is the value of the superconducting gap at the antinode ($\theta = 0, \pi/2$) [6]. This behavior is also apparent in studies based on electronic Raman scattering [7] and in determinations of the in-plane magnetic penetration depth λ_{ab} [8, 9].

Although the majority viewpoint in the high-T_c community seems to argue for such non-s-wave pairing symmetry there are compelling dissenting views, particularly work within the past few years, by Müller [10], Harshman [11], Klemm [12], and many others. In particular, Müller concludes that "...recent experiments probing the surface and bulk of cuprate superconductors [show that their] character is d on the surface and substantially s in the bulk." This conclusion has been bolstered by muon-spin relaxation (μSR) experiments with YBCO reported and interpreted in Ref.[11]. Several authors [13, 14] have proposed that the doping process could modify the electron-phonon interaction and the Fermi surface with a concomitant shift from d- to s-type coupling as doping increases. The strongest evidence for an s-wave order parameter in a cuprate is reviewed in Ref.[12] where several c-axis twist experiments on BSCCO along with earlier c-axis tunneling between BSCCO/Pb junctions are surveyed. Ref.[15] summarizes many of the problems with the so-called "phase-sensitive" tests [16] in YBCO. Additionally, predictions made in Ref.[17] that a vortex in a d-wave superconductor would exhibit a measurable density of states in a four-fold pattern emanating from the core have not been observed [18] in either YBCO or BSCCO. On the contrary, the vortex cores appear consistent [18] with isotropic s-waves.

Replacing the characteristic phonon-exchange Debye temperature $\Theta_D \equiv \hbar\omega_D/k_B$ of around $400K$ (with k_B the Boltzmann constant and ω_D the Debye frequency) by a characteristic *magnon*-exchange temperature of around $1000K$ can lead to a simple model interaction such as the BCS one but associated with spin-fluctuation-mediated pairing [19].

The so-called "Uemura plot" [20] of data from μSR, neutron and Raman scattering, and ARPES measurements exhibit T_c *vs* Fermi temperatures $T_F \equiv E_F/k_B$ where E_F the Fermi energy and k_B the Boltzmann constant. Empirical T_cs of many cuprates straddle a straight line *parallel* to the Uemura-plot line associated with the simple BEC formula $T_B \simeq 3.31\hbar^2 n_B^{2/3}/m_B k_B \simeq 0.218 T_F$ corresponding to an ideal gas of bosons of mass $m_B = 2m^*$ and number density $n_B = n_s/2$ where m^* is the effective mass and n_s the number density of individual charge carriers. The parallel line is shifted down from T_B by a factor 4-5. This has been judged [21] as a "fundamental importance of the BEC concept in cuprates."

Previous theoretical papers on the possible origin of HTSC [14, 22–24] proposed that the phenomenon might be rooted in a 2D Bose-Einstein condensate (BEC) of CPs pre-existing above T_c and coupled through a BCS-like phonon mechanism [25], originally taken as s-wave. As apparently first reported by Schrieffer [26], the Cooper

model interaction [27] leads to an approximate *linear* energy-vs-CMM relation $\frac{1}{2}v_F\hbar K$ for excited CPs propagating in the Fermi sea in 3D. This kind of dispersion relation is not unique to the Cooper model interaction. For example, an attractive interfermion delta potential [28] in 2D (imagined regularized [29] to support a single bound state of binding energy $B_2 \geqslant 0$) leads [30] (for a review see Ref.[24]) to the general CP dispersion relation $\mathcal{E}_K = \mathcal{E}_0 + c_1\hbar K + \left[1-(2-16/\pi^2)E_F/B_2\right]\hbar^2K^2/4m^* + O(K^3)$, where $c_1 = 2v_F/\pi$ precisely as with the Cooper model interaction [31]. Hence, the leading-order linearity is not induced by the particular interfermion interaction binding the CPs but is a consequence of the Fermi sea with $v_F \neq 0$ and in which a CP by definition propagates. Only in the vacuum limit $v_F \to 0 \Rightarrow E_F \equiv \frac{1}{2}m^*v_F^2 \to 0$ does that general dispersion relation reduce by inspection to the expected quadratic form $\mathcal{E}_K = \mathcal{E}_0 + \hbar^2K^2/4m^*$ for a composite object of mass $2m^*$. For either inter-electron interaction model, the linear term is a consequence of the presence of the Fermi sea. The formation of a BEC of CPs in 2D does not violate Hohenberg's theorem [32] as this holds only for quadratically-dispersive particles. The predicted 2D BEC temperature is $T_c \propto \left(n^{2D}\right)^{1/2} \propto \left(\Theta_D T_F\right)^{1/2}$ where n^{2D} is the CP number per unit area. This leads to values of T_c that are too high compared with empirical values. However, these schemes provide a correct description of other relevant physical properties of HTSCs such as a short coherence length, a type II magnetic behavior, and the temperature dependence of the electronic heat capacity [14]. They also lead to excellent fits of the condensate fraction curves for quasi-2D cuprates just below T_c [33], as well as for 3D and even quasi-1D SCs. To go beyond the simple s-wave interaction, an l-wave formulation of BCS theory was discussed by Schrieffer [26] himself and studied in considerable detail by Anderson and Morel [34] in the weak-coupling limit. This has been successfully employed [6, 35] to describe thermodynamic and transport properties of high-T_c cuprates. The d-wave extension in strong-coupling Eliashberg theory is reported in Refs.[36].

Here we develop a general l-wave BCS-type theory which is then applied in a quasi-2D BEC picture with either $l = 0$ or $l = 2$ pairing symmetry. In § II the l-wave BCS theory within the framework of the present model is discussed. In § III we study a quasi-2D BEC of linearly-dispersive, massless-like CPs and we evaluate the number density. In § IV the areal density n^{2D} of charge carriers is estimated by calculating the magnetic penetration depth arising from the CPs. In § V an analytical expression for the critical BEC temperature is derived, which is then applied in § VI for various superconducting materials including YBCO under different doping levels. Discussion and conclusions are given in § VII.

II. BCS THEORY WITH l-WAVE PAIRS

Some aspects of the l-wave BCS theory [6, 26, 34] relevant to our HTSC model follow. Consider a system of electron- (or hole-) pairs formed via a two-fermion isotropic potential V near the Fermi surface and with kinetic energies $\epsilon_k \equiv \hbar^2(k^2 - k_F^2)/2m^*$ (with $\hbar k_F$ the Fermi momentum) taken relative to the Fermi energy. The Pauli principle prevents background fermions in electron states just below (above) the Fermi surface from participating in the interaction. In the absence of external forces each pair propagates freely within a layer of finite width δ along the z direction and infinite extent on the $x-y$ plane so that its total momentum $\hbar\mathbf{K} = (\hbar\mathbf{K}_\parallel, \hbar K_z)$ is a constant of motion. By neglecting spin-dependent interactions the total spin S is conserved too and for a spin singlet $S = 0$ configuration the Pauli principle requires that the orbital wavefunction be of the form $\Psi(\mathbf{r}_1, \mathbf{r}_2) = \exp(i\mathbf{K}_\parallel \cdot \mathbf{R}_\parallel)\cos(K_z z)\Phi(\mathbf{r})$, where the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $\mathbf{R}_\parallel$ is the horizontal projection of the CM coordinate $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$, and $K_z = n\pi/\delta$ (with n integer). The z-dependence of the wavefunction ensures that the vertical flux of the electron (hole) pair across the layer boundary is null. Since the relative-coordinate problem is isotropic then $\Phi(\mathbf{r})$ is an eigenfunction of angular momentum with quantum numbers $l = 0, 1, 2, \cdots$. The total spin $S = 0$ singlet eigenstates of the system satisfy the Schrödinger equation

$$(H_0 - V)\Psi(\mathbf{r}_1, \mathbf{r}_2) = \mathcal{E}\Psi(\mathbf{r}_1, \mathbf{r}_2) \tag{1}$$

where H_0 is the free Hamiltonian V the interaction potential and $\mathcal{E}$ the energy eigenvalue. For a given CMM wavevector $\mathbf{K}$, we may expand the wave function as

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \exp(i\mathbf{K}_\parallel \cdot \mathbf{R}_\parallel)\cos(K_z z)\sum_{\mathbf{k}} a_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{r}). \tag{2}$$

In momentum space (1) thus becomes

$$\left(\mathcal{E} - \epsilon_{\mathbf{k}+\mathbf{K}/2} - \epsilon_{\mathbf{k}-\mathbf{K}/2}\right) a_{\mathbf{k}} = \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} a_{\mathbf{k}'} \tag{3}$$

with $V_{\mathbf{k}\mathbf{k}'} = <\mathbf{k}, -\mathbf{k}|V|\mathbf{k}', -\mathbf{k}'>$. Since the interaction potential V depends only on $\mathbf{r}$ it admits the expansion

$$V_{\mathbf{k}\mathbf{k}'} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} V_l(|\mathbf{k}|, |\mathbf{k}'|) Y_l^m(\Omega_{\mathbf{k}}) Y_l^{-m}(\Omega_{\mathbf{k}'}). \tag{4}$$

For small coupling amplitudes $V_l(|\mathbf{k}|, |\mathbf{k}'|)$ the contributions of different l spherical harmonics $Y_l^m(\Omega_{\mathbf{k}})$ in (4) can with good accuracy be considered relatively independent [34]. In that case (3) yields an analytical solution by assuming that V_l is separable, i.e., $V_l(|\mathbf{k}|, |\mathbf{k}'|) = V_0^{(l)} f_k^l f_{k'}^{l*}$ so that (3) becomes

$$\left(\mathcal{E} - \epsilon_{\mathbf{k}+\mathbf{K}/2} - \epsilon_{\mathbf{k}-\mathbf{K}/2}\right) a_{\mathbf{k}} = V_0^{(l)} f_k^l \sum_{k'} a_{k'} f_{k'}^{l*} \tag{5}$$

where $a_{\mathbf{k}} = a_k Y_l^m(\Omega_{\mathbf{k}})$. Eq.(5) can be now rewritten as

$$a_k^{(l)} = C^{(l)} \frac{V_0^{(l)} f_k^l}{\mathcal{E}_K^{(l)} - \epsilon_{\mathbf{k}+\mathbf{K}/2} - \epsilon_{\mathbf{k}-\mathbf{K}/2}} \tag{6}$$

where $\sum_{\mathbf{k}'} a_{k'} f_{k'}^{l*} \equiv C^{(l)}$ is a constant. One thus obtains a BCS-type integral relation for a CP in the eigenstate characterized by (l, m)

$$1 = V_0^{(l)} \sum_k \frac{|f_k^l|^2}{\mathcal{E}_K^{(l)} - \epsilon_{\mathbf{k}+\mathbf{K}/2} - \epsilon_{\mathbf{k}-\mathbf{K}/2}}. \tag{7}$$

Following Schrieffer [26] we assume that the angular-independent l component of the generalized BCS model interaction (4) is given by

$$V_0^{(l)} f_k^l f_{k'}^{l*} = -V_0 \tag{8}$$

with $V_0 > 0$ for CPs with relative momenta $(\mathbf{k}, \mathbf{k}')$ lying in the neighborhood of the Fermi surface

$$k_F < |\mathbf{k} + \mathbf{K}/2|, |\mathbf{k} - \mathbf{K}/2| < K_{max} \tag{9}$$

and $V_0^{(l)} f_k^l f_{k'}^{l*} = 0$ otherwise. Here $K_{max} = \sqrt{k_F^2 + k_D^2}$ with k_D defined in terms of the Debye energy via $\hbar\omega_D \equiv \hbar^2 k_D^2/2m^*$. A straightforward analysis [26] reveals that (7) yields a bound state with energy $\mathcal{E}^{(l)} < 0$ for arbitrarily weak coupling so long as the potential is attractive in the region (9) in k-space. Then, a bosonic CP can form only if the tip of vector $\mathbf{k}$ lies within the intersection of the two spherical shells defined by (9) whose center-to-center separation is K; fermions with wave vectors lying outside this overlap are unpairable [24].

In the quasi-2D limit the fundamental expression (7) can be evaluated by substituting the summation over $\mathbf{k}$ by a 2D integration. In addition, for small δ the only term in K_z that yields a finite contribution is $n = 0$. By assuming a 2D cylindrical Fermi surface we obtain

$$\begin{aligned} 1 &= \frac{V_0}{(2\pi)^2} \int_0^{2\pi} d\theta \int_{k_1}^{k_2} \frac{k dk}{|\mathcal{E}_K^{(l)}| + \epsilon_{\mathbf{k}+\mathbf{K}/2} + \epsilon_{\mathbf{k}-\mathbf{K}/2}} \\ &\simeq \frac{m^* V_0}{4\pi} \Big\langle \ln \Big| \frac{|\mathcal{E}_K^{(l)}| + 2\hbar\omega_D - v_F \hbar K \cos\theta}{|\mathcal{E}_K^{(l)}| + v_F \hbar K \cos\theta} \Big| \Big\rangle_F \end{aligned} \tag{10}$$

where $k_1 = k_F + (K/2)\cos\theta$, $k_2 = k_F + k_D - (K/2)\cos\theta$, and the approximate equality in the second row holds up to terms of order $(k_D/k_F)^2 \equiv \Theta_D/T_F$. The angular brackets denote an average over a 2D cylindrical Fermi surface $< \ldots >_F \to (1/2\pi)\int_0^{2\pi} d\theta$. The Fermi average then gives the energy spectrum of excited CPs [26]:

$$\mathcal{E}_K^{(l)} \simeq \mathcal{E}_0^{(l)} + c_1 \hbar K + O(K^2) \tag{11}$$

where $c_1 \equiv 2v_F/\pi$ in 2D and $\mathcal{E}_0^{(l)}$ is the binding energy of the CP ground state ($\hbar K = 0$) [27]

$$\mathcal{E}_0^{(l)} = -2\hbar\omega_D / [\exp(2/\mathcal{N}_0 V_0^{(l)}) - 1]. \tag{12}$$

The dispersion relation (11) is linear in leading order rather than quadratic as would be expected *in vacuo*. As a consequence, all *excited* CPs behave like a gas of free massless-like bosons with a common group velocity $c_1 = \hbar^{-1} d\mathcal{E}_K^{(l)}/dK$, but a variable energy determined by their CMM $\hbar K$. The dispersion relation (11) implies that in order for a CP to remain bound (i.e., $\mathcal{E}_K^{(l)} < 0$) its maximum CMM wavenumber must not exceed the value $|\mathcal{E}_0^{(l)}|/c_1 \equiv K_0$ since CPs with $K > K_0$ have $\mathcal{E}_K^{(l)} > 0$ and thus break up [26].

Explicit expressions of relevant thermodynamic variables and transport coefficients evaluated within the weak-coupling limit of the l-wave BCS theory have been derived in Refs.[6, 34, 35]. In these papers it is shown that the *average* behavior of most of these quantities over the cylindrical Fermi surface exhibits small variation due to the explicit realization of an $l = 0$ or $l = 2$ symmetry [6, 35]. In particular, the temperature-dependent gap equation is given by [6, 26, 34]

$$1 = \mathcal{N}_0 V_0 \int_0^{\hbar\omega_D} d\epsilon_{\mathbf{k}} \Big\langle \frac{\tanh\left(\frac{1}{2}\beta\sqrt{\epsilon_{\mathbf{k}}^2 + \Delta^{(l)2}|g^{(l)}(\mathbf{k})|^2}\right)}{\sqrt{\epsilon_{\mathbf{k}}^2 + \Delta^{(l)2}|g^{(l)}(\mathbf{k})|^2}} \Big\rangle_F \tag{13}$$

where $\beta = 1/k_B T$, $g^{(0)}(\mathbf{k}) = 1$ for $l = 0$, and $g^{(2)}(\mathbf{k}) = \cos(2\theta)$ for $l = 2$. The critical temperature is determined from (13) by the condition $\Delta^{(l)}(T_c) = 0$. In the weak-coupling limit $\hbar\omega_D/k_B T_c \gg 1$ it can be calculated analytically and it follows that T_c is independent of the $l-$state [34]:

$$k_B T_c = \frac{2e^\gamma}{\pi} \hbar\omega_D \exp(-1/\mathcal{N}_0 V_0) \simeq 1.13 \hbar\omega_D \exp(-1/\mathcal{N}_0 V_0) \tag{14}$$

with $\gamma \simeq 0.577\cdots$ Euler's constant. In the zero-temperature limit $\Delta_0^{(l)} \equiv \Delta^{(l)}(T = 0)$ the energy integration in (13) leads to the gap relation [34]

$$\Delta_0^{(l)} = 2\Gamma^{(l)} \hbar\omega_D \exp(-1/\mathcal{N}_0 V_0^{(l)}) \tag{15}$$

where $\Gamma^{(l)} = \exp[-\langle |g^{(l)}|^2 \ln |g^{(l)}| \rangle_F]$. For $l = 0$, $\Gamma^{(0)} = 1$, while for $l = 2$, $\Gamma^{(2)} = 2\exp(-1/2) \simeq 1.213$, so that combining (14) and (15) we are led to the gap-to-T_c ratios

$$\frac{2\Delta_0^{(0)}}{k_B T_c} \simeq 3.53 \qquad \frac{2\Delta_0^{(2)}}{k_B T_c} \simeq 4.28. \tag{16}$$

For $l = 0$ one recovers the standard BCS result [25] and the somewhat higher value for the $l = 2$ d-wave case. We note that the quantity $\Delta_0^{(l)}/\Gamma^{(l)}$ has the same functional dependence as the zero-temperature gap of the BCS theory [25]. Considering that measurements of the energy gap for any given cuprate show some scatter about a central value Δ_0^{exp} [38] in the following we shall assume that $\Delta_0^{exp} \simeq \Delta_0^{(2)}/\Gamma^{(2)} \simeq \Delta_0^{(0)} \equiv \Delta_0$.

On the other hand, the average superfluid density $\rho_s(T) \equiv \lambda_{ab}^2(0)/\lambda_{ab}^2(T)$ exhibits a more pronounced an-

gular momentum dependence. This is given by [6]

$$\rho_s^{(l)} = 1-\beta\int_0^\infty d\epsilon_{\mathbf{k}}\Big\langle \frac{|g^{(l)}(\mathbf{k})|^2}{\cosh^2\left(\frac{1}{2}\beta\sqrt{\epsilon_{\mathbf{k}}^2+\Delta^{(l)2}|g^{(l)}(\mathbf{k})|^2}\right)}\Big\rangle_F. \quad (17)$$

In the low-temperature limit (17) yields for $l = 0$ an exponential T-dependence

$$\rho_s^{(0)}(T) \simeq 1 - \left(\frac{2\pi\Delta_0^{(0)}}{k_BT}\right)^{1/2}\exp(-\Delta_0^{(0)}/k_BT) \quad (18)$$

while for $l = 2$ it gives the linear T-dependence

$$\rho_s^{(2)}(T) \simeq 1 - \frac{(2\ln 2)k_BT}{\Delta_0^{(2)}}. \quad (19)$$

Experiments [8, 9] on the temperature variation of the magnetic penetration depth $\lambda_{ab}(T)$ are consistent with the quasi-linear behavior (19) which is a signature of d-wave symmetry. However, its asymptotic value $\lambda_{ab}(T \to 0)$ is independent of l, a result that we apply below.

III. BOSE-EINSTEIN CONDENSATION

We assume that charge carriers are an ideal binary mixture of non-interacting unpaired fermions plus breakable bosonic linearly-dispersive CPs [14, 22, 23, 26]. Let the fermion number per unit area be $n_f = n_{f1} + n_{f2}$ where n_{f1} and n_{f2} are the number densities of unpairable and pairable fermions, respectively. Unpairable fermions lie outside the interaction region of (8) unlike the pairable fermions whose T-dependent density $n_{f2}(T)$ is

$$n_{f2}(T) = 2\big[\, n_0^{2D}(T) + n_{0<K\le K_0}^{2D}(T)\,\big] + n_{f2}^u(T). \quad (20)$$

Here n_0^{2D} is the bosonic number density of CPs with CMM wavenumber $K = 0$, $n_{0<K<K_0}^{2D}$ that with $0 < K < K_0$, and n_{f2}^u the number density of pairable but unpaired fermions. By asserting that in thermal equilibrium these kinds of fermions arise precisely from broken CPs [22] we identify $n_{f2}^u(T) = 2n_{K_0<K<K_{max}}^{2D}(T)$. On the other hand, at $T = 0$ all pairable fermions should belong to the condensate (Ref.[38], p. 122) so that $n_{f2}(0) = 2n_0^{2D}(0) \equiv 2n^{2D}$ where n^{2D} is the total boson number per unit area. The number equation for pairable fermions may thus be reexpressed in terms of boson quantities alone, namely $n^{2D} = n_0^{2D}(T)+n_{0<K\le K_0}^{2D}(T)+n_{K_0<K\le K_{max}}^{2D}(T)$ $\equiv n_0^{2D}(T) + n_{0<K\le K_{max}}^{2D}(T)$. Thus

$$n^{2D} = n_0^{2D}(T) + \int_{0^+}^{K_{max}} \frac{d^2K}{(2\pi)^2}\frac{1}{z^{-1}\exp\beta\mathcal{E}_K^{(l)} - 1} \quad (21)$$

where $\beta \equiv 1/k_BT$, μ the boson chemical potential and $z \equiv \exp\beta\mu$ is the fugacity ($0 \le z \le 1$). On introducing (21) the energy-shifted boson dispersion relation $\mathcal{E}_K^{(l)} = \hbar c_1 K$ for $K > 0$ the integral can evaluated by changing to the variable $x \equiv \beta\hbar c_1 K$. Since

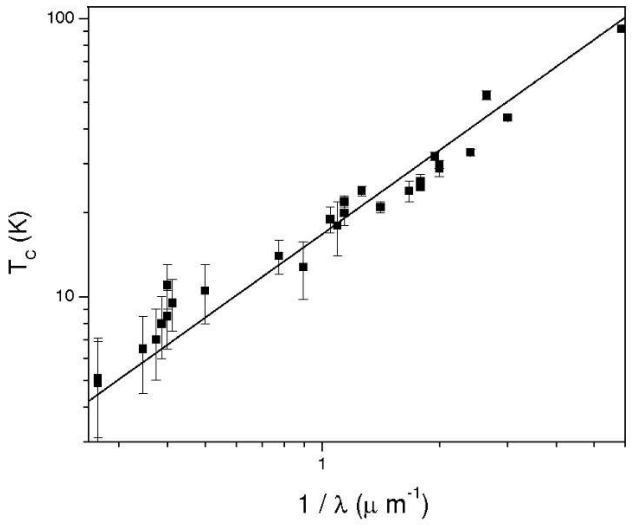

FIG. 1: Comparison of experimental T_cs vs. theoretical predictions (29) as function of zero-temperature inverse penetration length λ_{ab}^{-1} for YBCO compounds with different doping degrees. Square datapoints are taken from Ref.[37], except for uppermost square referring to the optimally doped regime [38]. Vertical "error bars" represent full widths of σ_1 peaks, where σ_1 is the real part of the conductivity σ employed in Ref.[37] to determine λ_{ab}^{-1}.

$c_1 = 2v_F/\pi$ and $K_{max} = k_F(1+k_D^2/k_F^2)$, the upper integration limit x_{max} in (21) is then be very large, namely $x_{max} = \beta\hbar v_F k_F = 2E_F/k_BT \gg 1$. The last inequality is consistent with the maximum empirical value for the ratio $k_BT_c/E_F \le 0.05$ reported [20] for all SCs including cuprate SCs. Given the rapid convergence of Bose integrals the upper integration limit x_{max} may safely be taken as infinite in (21) so that the integrals can then be evaluated exactly by expanding the integrand in powers of $z\exp(-x)$ and integrating term by term. The number density (21) becomes

$$n^{2D} = n_0^{2D}(T) + \frac{(k_BT)^2}{2\pi\hbar^2c_1^2}\sum_{n=1}^{\infty}\frac{z^n}{n^2} \quad (22)$$

The critical BEC temperature T_c is now determined by solving (22) for $n_0^{2D}(T_c) = 0$ and $z(T_c) = 1$. One obtains

$$T_c = \frac{\hbar c_1}{k_B}\left(\frac{2\pi n^{2D}}{\zeta(2)}\right)^{1/2} \quad (23)$$

where $\zeta(2) = \pi^2/6$.

IV. CHARGE CARRIER DENSITY

The areal density of charge carriers was formerly estimated from measurements of the London penetration

superconductor	Θ_D (K)^a	Δ_0 (meV)^b	λ_{ab} (nm)^c	$\delta(\AA)^d$	T_c^{exp}(K)^e	T_c^{th} (K)	$(2\Delta_0/k_BT_c)^{exp(f)}$	$(2\Delta_0/k_BT_c)^{th}$
$(La_{.925}Sr_{.075})_2CuO_4$	360	6.5	250	4.43^g	36	36.4	4.3	4.14
$YBa_2Cu_3O_{6.60}$	410	15.0	240	2.15^h	59	56.0	5.90	6.09
$YBa_2Cu_3O_{6.95}$	410	15.0	145	2.15^g	93.2	92.6	4.0	3.68
$Bi_2Sr_2CaCu_2O_8$	250	16.0	250	2.24^g	80	72.2	4.64	4.85
$Bi_2Sr_2Ca_2Cu_3O_{10}$	260	26.5	252	2.24^i	108	109.2	5.7	4.99
$Tl_2Ba_2Ca_2Cu_2O_8$	260	22.0	221	2.14^i	110	104.1	4.5	4.47
$Tl_2Ba_2Ca_2Cu_3O_{10}$	280	14.0	200	4.30^i	125	105.5	3.1	2.96

TABLE I: Physical parameters of cuprate superconductors and predicted values for T_c, and the ratio $2\Delta_0/k_BT_c$ according to (28). Debye temperature is $\Theta_D \equiv \hbar\omega_D/k_B$. Parameters taken from from Ref.[38] (see also references cited therein): a) table 4.1, b) table 6.1, c) table A.1, d) table A.2, e) table A.1, f) table 6.1, g) estimated from band-structure calculations [39, 40], h) estimated as $\delta = 0.64\ c_{int}$, and i) estimated as $\delta = 0.68\ c_{int}$, where c_{int} is the CuO_2 interlayer separation for a given cuprate. For YBCO Θ_D, Δ_0, and δ are assumed the same for different dopings.

depth λ_L which is the distance over which an external magnetic field decays within the superconductor. For superelectrons with a 3D density n_s, charge e, and effective mass m^*, one has the well-known relation $1/\lambda_L^2 = 4\pi e^2 n_s/m^*c^2$. By introducing [20] the average interlayer spacing c_{int} between CuO_2 planes in HTSCs it follows that $n^{2D} \simeq c_{int} n_s$. Penetration-depth data spanning a wide range of critical temperatures are consistent with the phenomenological Uemura relation $T_c \propto 1/\lambda_L^2 \propto n^{2D}/m^*$ [20].

Within the present model we evaluate the magnetic penetration depth due to linearly-dispersive CPs with charge $2e$, and constrained to move within a thin layer of width δ with a uniform CM speed c_1. Thus, we first consider the expression for the 3D supercurrent of excited CPs [14] $\mathbf{J}_s = n^{3D}(2e)c_1\hat{\mathbf{K}}$ with $\hat{\mathbf{K}} \equiv \mathbf{K}/K$. We now introduce the contour integral of the CP wavefunction phase within a homogeneous medium and in the presence of an external magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. The following integral along any closed path vanishes

$$\oint \left(\hbar\mathbf{K} + \frac{2e}{c}\mathbf{A}\right) \cdot d\mathbf{r} = 0 \tag{24}$$

where c is the speed of light in vacuum. By expressing $\mathbf{K}$ in terms of $\mathbf{J}_s$ and using Stoke's theorem to evaluate (24) we get a modified version of London's equation $\mathbf{J}_s = -\Lambda_p \mathbf{A}$ where $\Lambda_p \equiv 4e^2 c_1 n^{3D}/\hbar c K$. Taking now the curl of this modified London equation and introducing Ampere's law $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{J}_s$, it follows that the magnetic induction $\mathbf{B}$ satisfies the Helmholtz equation $\nabla^2\mathbf{B} = \lambda^{-2}\mathbf{B}$, where

$$\frac{1}{\lambda^2} \equiv \frac{(2e)^2}{c^2}\frac{4\pi c_1 n^{3D}}{\hbar K}. \tag{25}$$

Note that London's result is recovered for quasi-particles with density $n^{3D} \to n_s/2$, momentum $\hbar K \to 2m^*c_1$, and charge $2e \to e$. This expression for λ varies between its minimum value $\lambda = 0$ when $K = 0$ (perfect diamagnetism), and its maximum, say λ_0, for $K = K_0$ (CP breakup). It seems natural to identify λ_0 with the experimentally observed value of the in-plane penetration depth at $T = 0$, namely $\lambda_0 = \lambda_{ab}(T = 0)$. Here, $\lambda_{ab}^{-1}(0) = \lambda_a^{-1}(0) + \lambda_b^{-1}(0)$ is the geometric mean of this parameter measured along crystallographic in-plane directions a and b. As shown in § II, this parameter is independent of the explicit value of the angular momentum l. By substituting the dispersion relation (11) to eliminate K_0 from λ_0 and imposing the relation $n^{2D} = \delta n^{3D}$ the 2D charge carrier density becomes

$$n^{2D} = \frac{e^2}{c^2}\frac{\delta|\mathcal{E}_0^{(l)}|}{16\pi c_1^2}\frac{1}{\lambda_{ab}^2}. \tag{26}$$

This latter expression can be reformulated by considering the relation (15) and the weak-coupling limit of (12). It follows that $|\mathcal{E}_0^{(l)}| = (\Delta_0^{(l)})^2/2\hbar\omega_D$ so that

$$n^{2D} = \frac{e^2}{32\pi c_1^2 c^2}\frac{\delta\Delta_0^2}{\hbar\omega_D}\frac{1}{\lambda_{ab}^2} \tag{27}$$

where the approximate relation $\Delta_0^{(l)}/\Gamma^{(l)} \simeq \Delta_0$ as justified in § II was used.

V. CRITICAL TEMPERATURE

The final explicit expression for the critical BEC temperature T_c is now obtained by substituting (27) in (23). This leaves

$$T_c = \frac{\hbar c}{2\pi k_B e}\left(\frac{3\delta}{2\hbar\omega_D}\right)^{1/2}\frac{\Delta_0}{\lambda_{ab}} \tag{28}$$

which is independent of the CP speed c_1. We observe that for fixed values of ω_D, Δ_0, and δ, the critical temperature increases linearly with λ_{ab}^{-1}. This dependence has been observed by Zuev *et al.*[37] in experiments in underdoped YBCO films with T_cs ranging from 6 to $50K$. They conclude that, within some noise, their data fall on the same curve $\rho_s \propto \lambda_{ab}^{-2} \propto T_c^{2.3\pm0.4}$, irrespective of annealing procedure, oxygen content, etc. Thus, by assuming that except for λ_{ab} the other YBCO parameters are approximately independent of the doping level, we introduced in (28) the values: $\Theta_D = 410K$ [38], $\Delta_0 = 14.5$

meV [38], and $\delta = 2.15$ $\mathring{A}$ [39, 40] to get the relation

$$T_c = \frac{16.79[(\mu m)^{-1}K]}{\lambda_{ab}}. \tag{29}$$

Figure 1 is adapted from Ref.[37] and compares theoretical predictions (29) with experimental data, as well as with data pertaining to higher doping regimes. We see that (29) gives an excellent fit to the experimental data. The same functional dependence has been observed in single YBCO crystals near the optimally-doped regime [41]. More recently, Broun *et al*. [9] found that their samples of high-purity single-crystal YBCO followed the rule $T_c \propto \lambda_{ab}^{-1} \propto n_s^{1/2} \propto (p - p_c)^{1/2}$ where the doping p is the number of holes per copper atom in the CuO_2 planes and p_c the minimal doping for superconductivity onset. The measured value of the penetration length in YBCO crystals is an order of magnitude bigger than in thin films [9, 41], so that the specific values of T_cs derived from (28) are not in such good agreement as in the YBCO films. However, one should expect variations of parameters such as the energy gap associated to crystals and film systems. It has been pointed out [37] that YBCO *films* seem to behave more like other cuprates such as BiSrCaCuO or LaSrCuO than do YBaCuO *crystals*. Furthermore, a different approach [42] based on measurements of the lower critical magnetic field $H_{c1}(T)$ for highly underdoped YBCO indicates that experimental data may be consistently described only by assuming $T_c \propto n_s^{0.61}$, in close agreement with studies mentioned above.

Theoretical values of T_c for superconducting cuprates with different compositions have been also calculated using (28). Here we report on these seven layered-cuprate superconducting compounds: $(La_{.925}Sr_{.075})_2CuO_4$; $YBa_2Cu_3O_{6.60}$; $YBa_2Cu_3O_{6.95}$; $Tl_2Ba_2Ca_2Cu_2O_8$; $Tl_2Ba_2Ca_2Cu_3O_{10}$; $Bi_2Sr_2Ca_2Cu_3O_{10}$; and $Bi_2Sr_2CaCu_2O_8$. Characteristic parameters for these materials were taken from tables compiled in Ref.[38] (see also [43–45]). Concerning the layer width δ no direct experimental data are available. We have employed results derived from energy band-structure calculations for cuprates. Contour plots [39, 40] of the charge distribution for La_2CuO_4, $YBa_2Cu_3O_7$, and $BiCa_2SrCu_2O_8$ suggest that charge carriers in each of these systems are concentrated within slabs of average width $\delta \simeq 2.61\mathring{A}$, $2.15\mathring{A}$, and $2.28\mathring{A}$, respectively, about their CuO_2 planes. As c_{int} denotes the average separation between adjacent CuO_2 planes, it follows from crystallographic data [38] that the yttrium and bismuth compounds give $\delta \simeq 0.64\ c_{int}$ and $0.68\ c_{int}$, respectively. Taking into account that $BiSr_2Ca_nCu_{n+1}O_{6+n}$ compounds possess the same layering scheme as their $TlBa_2Ca_nCu_{n+1}O_{6+n}$ counterparts [38], we assumed that the condition $\delta \simeq 0.68 c_{int}$ holds also for the thallium compounds. The former estimations are congruent with Uemura's surmise [20] that SC charge carriers in layered cuprates are concentrated within slabs of width $\delta = c_{int}$.

Table I shows results obtained using the foregoing assumptions, together with the physical parameters involved in the calculation. In most cases we find rather satisfactory agreement between predicted and measured values of T_c. We also find very good agreement between theoretical and experimental gap-to-T_c ratios $2\Delta_0/k_BT_c$. Average theoretical and experimental such ratios presented in Table I are $(2\Delta_0/k_BT_c)^{th} \simeq 4.45$ and $(2\Delta_0/k_BT_c)^{exp} \simeq 4.59$, respectively. Both are consistent with the ratio $2\Delta_0^{(2)}/k_BT_c \simeq 4.28$ predicted by the $l = 2$ BCS theory in (16). We have not attempted estimate uncertainties of our theoretical results since the accumulated data of the physical parameters involved in the calculation, particularly Δ_0 and λ_{ab}, show a wide scatter.

VI. DISCUSSION AND CONCLUSIONS

We have shown that layered-cuprate HTSC can be described by means of an l-wave BCS theory for a quasi-2D BEC of Cooper pairs. The theory involves a *linear*, as opposed to quadratic, dispersion relation in their total or CM momenta. The theory yields a simple formula for the critical transition temperature T_c with a functional relation $T_c \propto 1/\lambda_{ab} \propto n_s^{1/2}$ which applies to a variety of cuprate SCs over a wide range of dopings. Although this behavior apparently disagrees with the phenomenological Uemura relation $T_c \propto 1/\lambda_{ab}^2$ [20], different experimental studies [9, 37, 41] show consistency with the inverse linear dependence of T_c. Additional consistency is also seen with the reported dependence $T_c \propto n_s^{0.61}$ arising from measurements of the lower critical magnetic field [42]. When averaged over a cylindrical Fermi surface, the physical quantities involved in the theory show small dependence on the angular momentum state l. However, the gap-to-T_c ratio $2\Delta_0/k_BT_c$ is closer to that predicted by the extended BCS theory for $l = 2$ than for $l = 0$. It is shown elsewhere [46] that all relevant 2D expressions derived here arise in the limit $k_BT\delta/\hbar c_1 \to 0$ of a more general 3D BCS-BEC theory for layered materials.

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